Handout Berlin 2008

Using technology to assess mathematics: Threat or treat?

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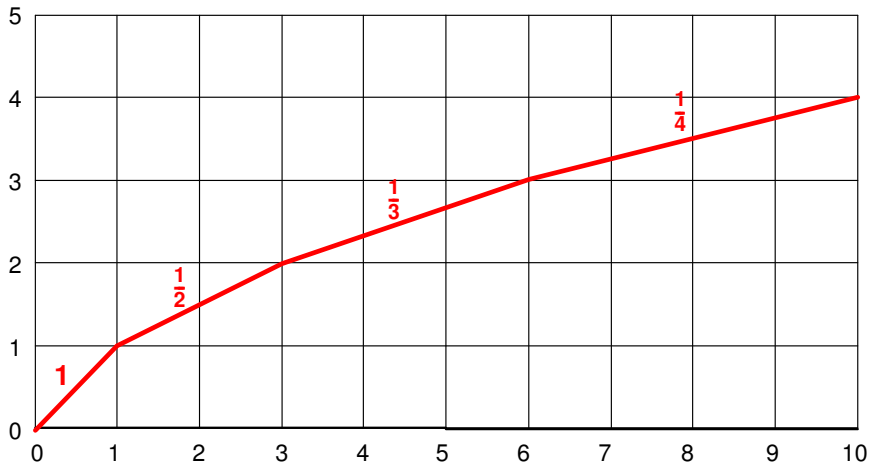
Problem 1
Slopes and Fractions

From the point (0, 0) a graph is drawn that exists of line segments with slopes $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots..$
The graph will be continued in this way up to and including a segment with slope $\frac{1}{10}$.

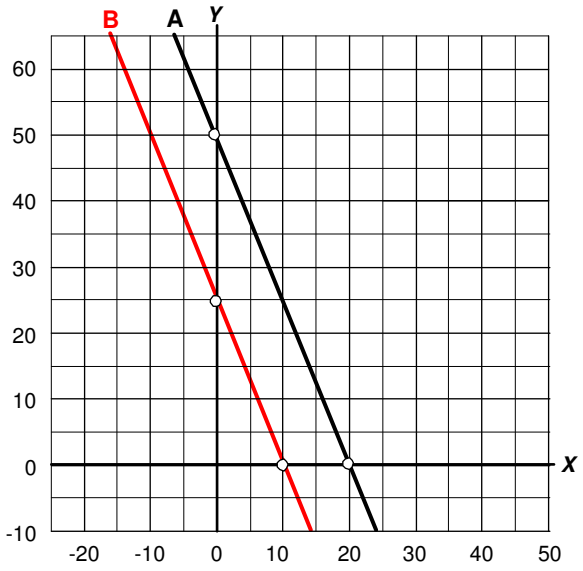


Draw this piecewise linear graph on your TI-Nspire.
What are the coordinates of the farthest point?
Show your work.

NB This problem was designed by Martin Kindt (Freudenthal Institute, The Netherlands) for the unit *Algebra Tools* of *Mathematics in Context.*

Problem 2
Non-intersecting graphs

Look at the graphs **A** and **B** in the diagram below.



How can you know for sure that the graphs **A** and **B** will not intersect, not even if the grid is extended?
Which special property does the difference graph **A - B** have?
Does having a graphing calculator available help to solve the problem?

NB This problem was designed by Martin Kindt (Freudenthal Institute, The Netherlands) for the unit *Algebra Rules!* of *Mathematics in Context.*

Problem 3

For every real $x \neq 0$, a function f is given by $f(x) = \frac{1}{2}x + \frac{x}{e^x - 1}$

Use a graphing calculator to draw the graph.
It looks as if the graph is symmetrical in the y-axis.
Prove that this is indeed the case.

Problem 4

For every real x, a function f is given by $f(x) = \frac{4x}{x^2 + 1}$

Use your graphing calculator to draw the graph for $y = f(x) - f(\frac{1}{x}) + 1$

What do you notice?

Use algebraic reasoning to support your statement.

Problem 5

Breathing

National exam vwo B1,2 (pre university course) period 1, year 2005

When your breathing is checked, you are asked to breathe out as far as you can and next take a deep breath and hold your breath for five seconds.
The amount of fresh air inhaled in the lungs during breathing in is a function of the time. For a healthy person, this model may be used for this relationship

$$L(t) = 3.61(1 - e^{-2.5t})$$

In this equation, L represents the amount of fresh air in liters and t is the time in seconds $(0 \le t \le 5)$

According to this model, the maximum amount of air inhaled by a healthy person is about 3.6 liters.

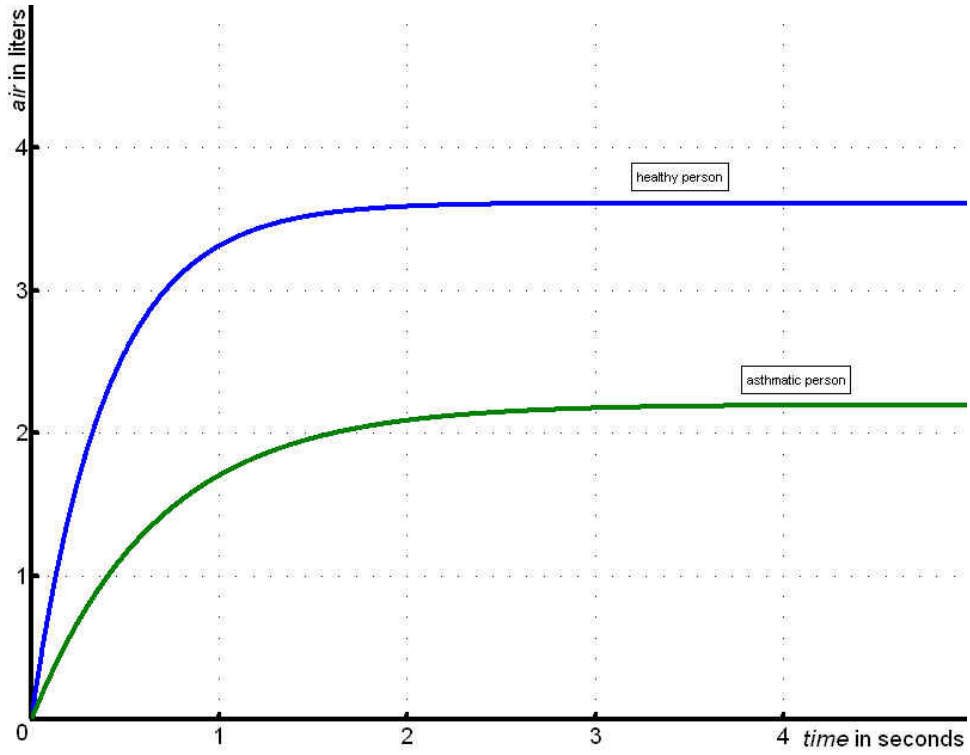
1. Use the model to find after how many seconds 90% of this maximum amount of fresh air has been inhaled.

Asthma is a respiratory ailment. For an asthmatic person, the maximum amount of fresh air inhaled in the lungs is smaller and it takes more time to reach this maximum. For an asthmatic person, the following model is suggested

$$L_\alpha(t) = \alpha \times 3.61(1 - e^{-2.5\alpha \times t})$$

In this equation, α is a constant between 0 and 1, depending on the severity of the asthmatic condition.

In the diagram below, the graph for a healthy person as well as a certain asthmatic person is shown.



2. For the graph of the asthmatic person, calculate the value of α, in one decimal. Show your work.

For the next problem, again use the formula $L_\alpha(t) = \alpha \times 3.61(1 - e^{-2.5\alpha \times t})$
The speed of the air being inhaled is at its maximum at $t = 0$.

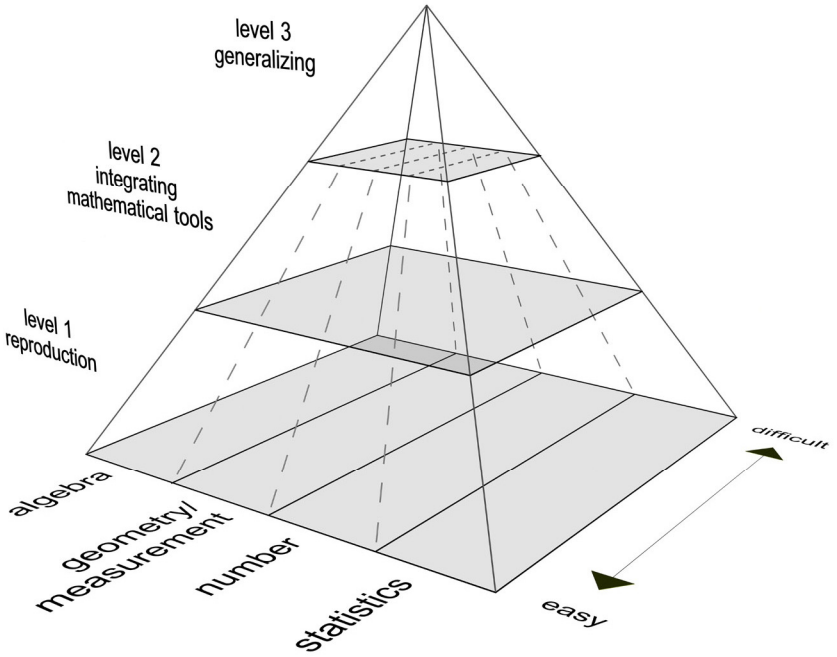
3. For which value of α is this maximum speed equal to 4.5 liters per second?

Test design characteristics

Some suggestions in arbitrary order

1. A full test should be 'balanced', which means that the competencies *reproduction*, *connections*, and *reflection* are addressed. (See Dutch pyramid model included)
2. A context used as problem situation should be as authentic as possible. Note that 'authentic' may involve fictitious elements such as the name of a country, the currency used, etcetera.
3. If a problem situation is used to ask mathematical questions, the starting point is the situation and not the mathematical content.
4. The questions asked within a context should be meaningful, the designer should ask herself "Why would anybody want to know?"
5. A 'rich' context enables mathematical questions from different domains. In general, limit yourself to one mathematical domain, e.g. algebra and use the context again for a different mathematical domain.
6. For less able students, the context should be closer to the student. For these students a scientific context is less suitable.
7. The language used must be adapted for the age and competencies of the students taking the test. This means for example that the text of a newspaper article used as a starting point for a (mathematical) question being posed almost always needs to be changed.
8. The first question to be posed within a context situation (if more than one question is posed) should enable students to get 'involved' in the context. Usually this will be a simple question at the reproduction level.
9. The number of questions to be used with one context is advised to be 3 – 5 to keep students motivated.
10. Make it clear whether or not the question posed should be solved within the situation or within the *mathematical model* of the situation. (For test purposes often a mathematical model of the situation is already provided).

The Dutch Assessment Pyramid

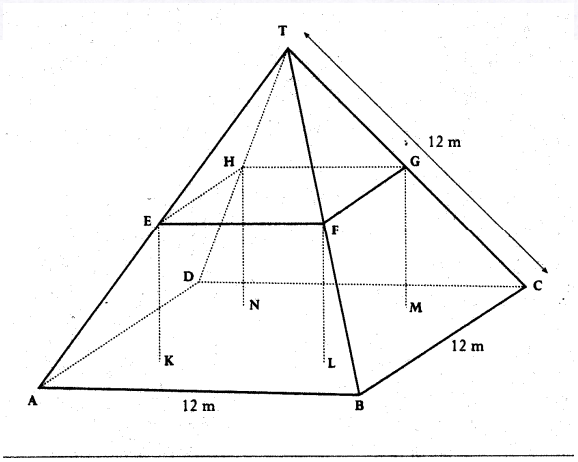


In a balanced test use level I : level II : Level III = 3 : 2 : 1

An example, based upon a PISA problem (M037), students aged 15:

A farmhouse has a roof in the shape of a pyramid.

When building the **roof** of such a farmhouse, the builders start by erecting four beams on the attic floor. Below is a student's mathematical model of the farmhouse roof with measurements added.



The attic floor, *ABCD* in the model, is a square.

1. Calculate the area of the attic floor. (Simple first question, reproduction competency)

The beams that support the roof are the edges of a block (rectangular prism) *KLMN EFGH*. The middle of *AT* is *E*, the middle of *BT* is *F*, *G* is the middle of *CT* and *H* is the middle of *DT*. All the edges of the pyramid shape in the model have length 12 m.

2. Find the length of *EF*, one of the edges of the block. (reproduction or connections)

When erecting the four beams on the attic floor (*EK*, *FL*, *GM*, and *HN* in the model) the builders need to find the proper place for these beams.

3. Make a scale drawing of the attic floor *ABCD* and show where *K*, *L*, *M*, and *N* should be placed. Explain how you found the place of these points and show any calculations you made. (reflection)

When building the roof, the builders were heard saying: "Let's put up the cube now." They meant they were going to erect the beams to form rectangular prism *KLMN EFGH*.

4. In this model, could rectangular prism *KLMN EFGH* be a cube? Explain why or why not. (reflection)

Possible student answers plus comments

Problem 1

Comment.

You may ask your students to draw the graph as shown using their GC ('learn to use'). The problem can not be solved using the GC, students have to reason mathematically to find the answer.

Answer

For each new line segment, a vertical step of one is taken, so you end at 10.

The horizontal steps are 1, 2, 3, 4, 5, ...10, so you end at

1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55

The farthest point is (55, 10)

Problem 2

Comment

Some students may try to find the functions and draw both graphs. In doing so, they also find the answer to the problem.

Answer

The slopes of both graphs are equal: $\frac{50}{-20} = \frac{25}{-10} = -2\frac{1}{2}$

This means that the graphs are parallel, and parallel lines do not intersect.

The graph corresponding with A – B is a horizontal line, $y = 25$. Have students check this by drawing the graph represented by $y_1 = 50 - 2\frac{1}{2}x$ and $y_2 = 25 - 2\frac{1}{2}x$.

Problem 3

Comment

Without the use of a graphic calculator, students would analyse the function and draw the graph. Next they need to demonstrate algebraically that $f(x) = f(-x)$ since they should know the property of a function that is symmetrical in the y-axis. This takes a lot of time. The analysis can now be done with the use of TI-Nspire.

Note that if you replace x by $-x$ in order to prove that $f(x) = f(-x)$ and enter $\frac{1}{2}x + \frac{x}{e^x - 1} = \frac{1}{2} \times (-x) - \frac{x}{e^{-x} - 1}$ in TI-Nspire, the answer **true** is shown, which of course is only an indication that you are on the right track.

If you draw both graphs, $f_1(x) = \frac{1}{2}x + \frac{x}{e^x - 1}$ and $f_2(x) = \frac{1}{2} \times (-x) - \frac{x}{e^{-x} - 1}$, you can see the graphs are equal though the functions are not, which again is an indication you are right. For many students this would suffice, it is not obvious for them that in order to *prove* the statement is true, you have to use algebraical tools.

Would you agree this problem is a difficult level I problem if students have studied the algebraical tools needed here, or is it a level II problem because they have to think of $f(x) = f(-x)$ first?

Answer

$f(x) = f(-x)$ if

$$\frac{1}{2}x + \frac{x}{e^x - 1} = \frac{1}{2} \times (-x) - \frac{x}{e^{-x} - 1}$$

$$\frac{xe^x - x + 2x}{2(e^x - 1)} = \frac{-x}{2} + \frac{x}{\frac{1}{e^x} - 1}$$

$$\frac{xe^x + x}{2(e^x - 1)} = \frac{-x}{2} + \frac{xe^x}{-1 + e^x}$$

$$\frac{xe^x + x}{2(e^x - 1)} = \frac{x - xe^x + 2xe^x}{2(e^x - 1)}$$

$$\frac{xe^x + x}{2(e^x - 1)} = \frac{xe^x + x}{2(e^x - 1)}$$

Problem 4

Comment

From the graph, one may conclude that $f(x) - f(\frac{1}{x}) = 0$, or $f(x) = f(\frac{1}{x})$. This needs to be proven algebraically. However, the graph does not show the exception for $x = 0$.

Answer

The graph shows $y = 1$

$$f(x) = \frac{4x}{x^2 + 1}$$

$$f(\frac{1}{x}) = \frac{4 \times \frac{1}{x}}{(\frac{1}{x})^2 + 1} = \frac{4}{x} \times \frac{x^2}{1 + x^2} = \frac{4x^2}{x(1 + x^2)} = \frac{4x}{1 + x^2}$$ (except for $x = 0$)

$y = \frac{4x}{x^2 + 1} - \frac{4x}{1 + x^2} + 1 = 1$ (except for $x = 0$) Check that y is undefined for $x = 0$ because of $\frac{1}{x}$.

Problem 5

(Score points are added from the original scoring guide).

Possible student answer	**score points**
1. $3.6(1 - e^{-2.5t}) = 0.9 \times 3.6$, or $1 - e^{-2.5t} = 0.90$	1
describe how solution is found, using a GC	1
$t \approx 0.92$, or $t \approx 0.9$	1
2. choose a point on the graph, for example (1; 1.7)	1

α can be found by solving the equation $\alpha \times 3.6(1-e^{-2.5\alpha}) = 1.7$ 1

describe how solution is found, using a GC

for example solve (1- $e^{(-2.5\times x)} = \frac{1.7}{3.6\times x}$, x) or use graphs and 'trace'

answer is α = 0.6 1

NB If for example the point on the graph (1; 1.6) was found, no score point is subtracted.

OR

the maximum value is 3.6α 1

from the graph it can be decided that 3.6α = 2.2 (or 2.1) 2

answer is α = 0.6

3. The speed of the air is $L'_\alpha(0)$ (the derivative in zero) 1

$L'_\alpha(t) = (\alpha \times 3.6 - 2.5^{-2.5\times\alpha}) \times -2.5\alpha =$ 2

$(= 9.0 \times \alpha^2 \times e^{-2.5\alpha t})$

$L'_\alpha(0) = 9.0 \times \alpha^2$ (liters per second) 1

$9.0 \times \alpha^2 = 4.5$ results in $\alpha \approx 0.71$ or $\alpha \approx 0.7$ 1